Scalars, Vectors and Tensors from Metric-Affine Gravity

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The metric-affine gravity provides a useful framework for analyzing gravitational dynamics since it treats metric tensor and affine connection as fundamentally independent variables. In this work, we show that, a metric-affine gravity theory composed of the invariants formed from non-metricity, torsion and curvature tensors decomposes exhaustively into a theory of scalar, vector and tensor fields. These fields are natural candidates for the ones needed by various cosmological phenomena. Indeed, we show that the model accommodates TeVeS gravity (relativistic modified gravity theory), vector inflation, and aether-like models. Detailed analyses of these and other phenomena can lead to a standard metric-affine gravity model encoding scalars, vectors and tensors necessitated by cosmology.

INTRODUCTION

Spacetime is a smooth manifold $M(g;\mathbb{\Gamma})$ endowed with a metric g and connection $\mathbb{\Gamma}$. Metric is responsible for measuring the distances while affine connection governs the straightness of curves and twirling of the manifold. These two geometrical structures, the metric and connection, are fundamentally independent geometrical variables, and they play completely different roles in spacetime dynamics. If they are to exhibit any relationship it derives from dynamical equations a *posteriori*. This fact gives rise to an alternative approach to Einstein's standard theory of general relativity: Metric-Affine Gravity.

The standard theory of general relativity is a purely metric theory of gravity since connection is completely determined by the metric and its partial derivatives, *a priori*. This determination is encoded in the Levi-Civita connection,

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\rho} \left(\partial_{\alpha} g_{\beta\rho} + \partial_{\beta} g_{\rho\alpha} - \partial_{\rho} g_{\alpha\beta}\right) \tag{1}$$

which defines a metric-compatible covariant derivative [1].

The metric-affine theory of gravity (similar to the Palatini formalism [2] in philosophy), which treats an metric tensor and connection as independent variables [1, 3], encodes a more general approach to gravitation by breaking up the a priori relation (1). This breaking inherently reveals the new dynamic structures torsion, nonmetricity in addition to curvature.

In this work we shall study metric-affine gravity in regard to decomposing the affine connection into independent vectors, tensors and scalars. We shall, in particular, be able to derive certain interactions using solely the *geometrical* sector with no reference to the matter sector that contains the known forces and species. Our starting point will be the fundamental independence of connection and metric, and the field content of the connection in the most general case.

The outline of the paper is as follows. In Sec. II below we first construct the most general 'connection' involving physically 'distinct and independent' structures, and then form a general action containing vector and tensor fields. In Sec. III we give specific applications of the derived action to vector inflation and TeVeS theory. Here we also discuss the relation of the model to the ones in the literature. In Sec. IV we conclude.

TENSOR-VECTOR THEORIES FROM NON-RIEMANNIAN GEOMETRY

An affine connection, whose components to be symbolized by $\mathbb{\Gamma}^{\lambda}_{\alpha\beta}$, governs parallel transport of tensor fields along a given curve in spacetime, and parallel transport around a closed curve, after one complete cycle, results in a finite mismatch if the spacetime is curved. Curving is uniquely determined by the Riemann curvature tensor

$$\mathbb{R}^{\mu}_{\alpha\nu\beta}\left(\mathbb{\Gamma}\right) = \partial_{\nu}\, \mathbb{\Gamma}^{\mu}_{\beta\alpha} - \partial_{\beta}\, \mathbb{\Gamma}^{\mu}_{\nu\alpha} + \mathbb{\Gamma}^{\mu}_{\nu\lambda}\mathbb{\Gamma}^{\lambda}_{\beta\alpha} - \mathbb{\Gamma}^{\mu}_{\beta\lambda}\mathbb{\Gamma}^{\lambda}_{\nu\alpha} \tag{2}$$

which is a tensor field made up solely of the non-tensorial objects $\mathbb{\Gamma}^{\lambda}_{\alpha\beta}$ and their partial derivatives. Notably, higher rank tensors involving $(n+1)$ partial derivatives of $\mathbb{\Gamma}^{\lambda}_{\alpha\beta}$ are given by n-th covariant derivatives of $\mathbb{R}^{\mu}_{\alpha\nu\beta}$, and hence, $\mathbb{R}^{\mu}_{\alpha\nu\beta}$ acts as the seed tensor field for a complete determination of the spacetime curvature.

Affine connection determines not only the curving but also the twirling of the spacetime. This effect is encoded in the torsion tensor

$$\mathbb{S}^{\lambda}_{\alpha\beta}\left(\mathbb{\Gamma}\right) = \mathbb{\Gamma}^{\lambda}_{\alpha\beta} - \mathbb{\Gamma}^{\lambda}_{\beta\alpha} \tag{3}$$

which participates in structuring of the spacetime together with curvature tensor. Torsion vanishes in geometries with symmetric connection coefficients, $\mathbb{\Gamma}^{\lambda}_{\alpha\beta} = \mathbb{\Gamma}^{\lambda}_{\beta\alpha}$.

The spacetime gets further structured by the notions of distance and angle if it is endowed with a metric tensor $g_{\alpha\beta}$ comprising clocks and rulers needed to make measurements. The connection coefficients and metric tensor are fundamentally independent quantities. They exhibit no *a priori* known relationship, and if they are to have any it must derive from some additional constraints. This property is best expressed by the non-metricity tensor

$$\mathbb{Q}^{\alpha\beta}_{\lambda}\left(g, \mathbb{\Gamma}\right) = \nabla^{\mathbb{\Gamma}}_{\lambda} g^{\alpha\beta} \tag{4}$$

which is non-vanishing for a general connection $\mathbb{\Gamma}^{\lambda}_{\alpha\beta}$. This rank (2,1) tensor would identically vanish if the connection were compatible with the metric. Indeed, in GR, for instance, the constraint to relate $\mathbb{\Gamma}^{\lambda}_{\alpha\beta}$ to $g_{\alpha\beta}$ is realized by imposing $\mathbb{\Gamma}^{\lambda}_{\alpha\beta}= \Gamma^{\lambda}_{\alpha\beta}$ from the scratch, where $\Gamma^{\lambda}_{\alpha\beta}$ is the Levi-Civita connection (1) which respect to which metric stays covariantly constant, $\nabla^{\Gamma}_{\lambda} g_{\alpha\beta} = 0$, and hence, non-metricity vanishes identically. Furthermore, for this particular connection, the torsion also vanishes identically since $\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha}$ by definition.

The curving and twirling of the spacetime are governed by the connection $\mathbb{\Gamma}^{\lambda}_{\alpha\beta}$. The metric tensor has nothing to do with them, and the Riemann curvature tensor (2) contracts, with no involvement of the metric tensor, in three different ways to generate the associated Ricci tensors of $\mathbb{\Gamma}^{\lambda}_{\alpha\beta}$:

- $\mathbb{R}_{\alpha\beta}\left(\mathbb{\Gamma}\right) \equiv \mathbb{R}^{\mu}_{\alpha\mu\beta}\left(\mathbb{\Gamma}\right),$
- $\widehat{\mathbb{R}}_{\alpha\beta}\left(\mathbb{\Gamma}\right) \equiv \mathbb{R}^{\mu}_{\alpha\beta\mu}\left(\mathbb{\Gamma}\right) = -\mathbb{R}_{\alpha\beta}\left(\mathbb{\Gamma}\right),$
- $\overline{\mathbb{R}}_{\alpha\beta}\left(\mathbb{\Gamma}\right) \equiv \mathbb{R}^{\mu}_{\mu\alpha\beta}\left(\mathbb{\Gamma}\right) = \partial_{\alpha}\, \mathbb{\Gamma}^{\mu}_{\beta\mu} - \partial_{\beta}\, \mathbb{\Gamma}^{\mu}_{\alpha\mu}.$

The reason for having more than one Ricci tensor is that the Riemann tensor (2) possesses only a single symmetry $\mathbb{R}^{\mu}_{\alpha\nu\beta}\left(\mathbb{\Gamma}\right) = -\mathbb{R}^{\mu}_{\alpha\beta\nu}\left(\mathbb{\Gamma}\right)$. It is this symmetry property that gives the relation $\widehat{\mathbb{R}}_{\alpha\beta}\left(\mathbb{\Gamma}\right) = -\mathbb{R}_{\alpha\beta}\left(\mathbb{\Gamma}\right)$ between the first two Ricci tensors above. The third Ricci tensor $\overline{\mathbb{R}}_{\alpha\beta}\left(\mathbb{\Gamma}\right)$ does not exist in the General Relativity (GR) since symmetries of the Riemann tensor, $\mathbb{R}_{\mu\alpha\nu\beta}\left(\Gamma\right) \equiv g_{\mu\mu'}\mathbb{R}^{\mu'}_{\alpha\nu\beta}\left(\Gamma\right) = -\mathbb{R}_{\mu\alpha\beta\nu}\left(\Gamma\right) = -\mathbb{R}_{\alpha\mu\nu\beta}\left(\Gamma\right) = \mathbb{R}_{\nu\beta\mu\alpha}\left(\Gamma\right)$, admits only one single independent Ricci tensor, the $\mathbb{R}_{\alpha\beta}\left(\Gamma\right)$ defined above.

Unlike the Riemann and Ricci tensors, the curvature scalar is obtained only by contraction with the inverse metric. Therefore, one finds the curvature scalar

$$\mathbb{R}\left(g,\mathbb{\Gamma}\right) \equiv g^{\alpha\beta}\mathbb{R}_{\alpha\beta}\left(\mathbb{\Gamma}\right) = -g^{\alpha\beta}\widehat{\mathbb{R}}_{\alpha\beta}\left(\mathbb{\Gamma}\right) \equiv -\widehat{\mathbb{R}}\left(g,\mathbb{\Gamma}\right) \quad (5)$$

from the first two Ricci tensors listed above. Likewise, the third Ricci tensor contracts to

$$\overline{\mathbb{R}}\left(g,\mathbb{\Gamma}\right) = g^{\alpha\beta}\overline{\mathbb{R}}_{\alpha\beta}\left(\mathbb{\Gamma}\right) = 0 \quad (6)$$

as dictated by the anti-symmetric nature of $\overline{\mathbb{R}}_{\alpha\beta}\left(\mathbb{\Gamma}\right)$. As a result, the theory possesses two distinct Ricci tensors but a single Ricci scalar.

The action density describing matter and gravity is formed by invariants generated by the tensor fields above plus the matter Lagrangian. A partial list includes

$$\begin{aligned}
&\mathbb{R},\ \mathbb{S}\bullet\mathbb{S},\ \mathbb{Q}\bullet\mathbb{Q},\ \mathbb{Q}\bullet\mathbb{S},\\
&\mathbb{R}^{2},\ \mathbb{R}\bullet\mathbb{R},\ \overline{\mathbb{R}}\bullet\overline{\mathbb{R}},\\
&\mathbb{R}\bullet\mathbb{S}\bullet\mathbb{S},\ \mathbb{R}\bullet\mathbb{Q}\bullet\mathbb{Q},\ \mathbb{R}\bullet\mathbb{Q}\bullet\mathbb{S},\\
&\overline{\mathbb{R}}\bullet\mathbb{S}\bullet\mathbb{S},\ \overline{\mathbb{R}}\bullet\mathbb{Q}\bullet\mathbb{Q},\ \overline{\mathbb{R}}\bullet\mathbb{Q}\bullet\mathbb{S},\\
&\mathbb{S}\bullet\mathbb{S}\bullet\mathbb{S}\bullet\mathbb{S},\ \mathbb{Q}\bullet\mathbb{Q}\bullet\mathbb{Q}\bullet\mathbb{Q},\\
&\mathbb{S}\bullet\mathbb{Q}\bullet\mathbb{Q}\bullet\mathbb{Q},\ \mathbb{S}\bullet\mathbb{S}\bullet\mathbb{Q}\bullet\mathbb{Q},\\
&\mathbb{S}\bullet\mathbb{S}\bullet\mathbb{S}\bullet\mathbb{Q},\ \mathcal{L}_{\tt matter}\left(g,\mathbb{\Gamma},\psi\right)
\end{aligned} \quad (7)$$

where $\mathcal{L}_{\tt matter}\left(g,\mathbb{\Gamma},\psi\right)$ is the Lagrangian of matter and radiation fields, which are collectively denoted by ψ. The first line of the list consists of mass dimension-2 invariants while the rest involve mass dimension-4 ones. Those structures having mass dimension-5 or higher are not shown. Also not shown are the invariants involving the covariant derivatives of the tensors. The bullet ($\bullet$) stands for contraction of the tensors in all possible ways by using the metric tensor, in case needed.

The scalars in (7), most of which do not exist at all in the GR, contain novel degrees of freedom reflecting the non-Riemannian nature of the underlying geometry. These degrees of freedom can be explicated via the decomposition of the connection

$$\mathbb{\Gamma}^{\lambda}_{\alpha\beta}= \Gamma^{\lambda}_{\alpha\beta} + \Delta^{\lambda}_{\alpha\beta} \quad (8)$$

with respect to the Levi-Civita connection (1), which is the most natural connection one would consider in the presence of the metric tensor. In this decomposition, $\Delta^{\lambda}_{\alpha\beta}$, being the difference between two connections, is a rank (1,2) tensor field, and it is the source of various non-Riemannian invariants listed in (7). To this end, in response to (8), the Ricci curvature tensor $\mathbb{R}_{\alpha\beta}\left(\mathbb{\Gamma}\right)$ splits as

$$\mathbb{R}_{\alpha\beta}\left(\mathbb{\Gamma}\right) = R_{\alpha\beta}\left(\Gamma\right) + \mathcal{R}_{\alpha\beta}\left(\Delta\right) \quad (9)$$

where $R_{\alpha\beta}\left(\Gamma\right) \equiv \mathbb{R}_{\alpha\beta}\left(\Gamma\right)$ is the Ricci curvature tensor of the Levi-Civita connection, and

$$\mathcal{R}_{\alpha\beta} = \nabla_{\mu}\Delta^{\mu}_{\beta\alpha} - \nabla_{\beta}\Delta^{\mu}_{\mu\alpha} + \Delta^{\mu}_{\mu\nu}\Delta^{\nu}_{\beta\alpha} - \Delta^{\mu}_{\beta\nu}\Delta^{\nu}_{\mu\alpha} \quad (10)$$

where $\nabla_{\alpha} \equiv \nabla^{\Gamma}_{\alpha}$ is the covariant derivative of the Levi-Civita connection $\Gamma^{\lambda}_{\alpha\beta}$. This tensor is a rank (0,2) tensor field generated by the tensorial connection $\Delta^{\lambda}_{\alpha\beta}$ alone. It is actually not a true curvature tensor as it is generated by none of the covariant derivatives $\nabla^{\mathbb{\Gamma}}$ or ∇^{Γ}. It is a 'quasi' curvature tensor.

In response to (8), the purely non-Riemannian Ricci tensor $\overline{\mathbb{R}}_{\alpha\beta}\left(\mathbb{\Gamma}\right)$ takes the form

$$\overline{\mathbb{R}}_{\alpha\beta}\left(\mathbb{\Gamma}\right) \ = \ \partial_{\alpha}{\tt V}_{\beta} - \partial_{\beta}{\tt V}_{\alpha} = \nabla^{\Gamma}_{\alpha}{\tt V}_{\beta} - \nabla^{\Gamma}_{\beta}{\tt V}_{\alpha} \quad (11)$$

wherein the second equality, which ensures that $\mathbb{R}_{\alpha\beta}\left(\mathbb{\Gamma}\right)$ is a rank (0,2) anti-symmetric tensor field, follows from the symmetric nature of the Levi-Civita connection, $\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha}$. It is obvious that $\overline{\mathbb{R}}_{\alpha\beta}\left(\mathbb{\Gamma}\right)$, in the form (11), is nothing but the field strength tensor

$$\overline{\mathbb{R}}_{\alpha\beta}\left(\mathbb{\Gamma}\right) \equiv {\tt V}^{(-)}_{\alpha\beta} \equiv \partial_{\alpha}{\tt V}_{\beta} - \partial_{\beta}{\tt V}_{\alpha} \quad (12)$$

of the Abelian vector

$${\tt V}_{\alpha} = \Delta^{\mu}_{\alpha\mu} \quad (13)$$

which is of purely geometrical origin. Consequently, purely non-Riemannian curvature tensor $\overline{\mathbb{R}}_{\alpha\beta}\left(\mathbb{\Gamma}\right)$ plays

a strikingly different role compared to $\mathbb{R}_{\alpha\beta}(\between)$ in that it directly extracts a vector field out of the underlying geometry.

As a result of (8), the torsion and non-metricity tensors

$$\mathbb{S}^{\lambda}_{\alpha\beta}(\between) \;=\; \Delta^{\lambda}_{\alpha\beta} - \Delta^{\lambda}_{\beta\alpha} \tag{14}$$
$$\mathbb{Q}^{\alpha\beta}_{\lambda}(g,\between) \;=\; \Delta^{\alpha}_{\lambda\mu} g^{\mu\beta} + \Delta^{\beta}_{\lambda\mu} g^{\alpha\mu} \tag{15}$$

reduce to plain algebraic expressions in terms of $\Delta^{\lambda}_{\alpha\beta}$.

Having explicated the $\Delta^{\lambda}_{\alpha\beta}$ dependencies of the fundamental tensor fields, it is time to ask what the tensorial connection actually is and what information about the geometry can be extracted from it. In other words, $\Delta^{\lambda}_{\alpha\beta}$, which embodies non-Riemannian ingredients of the underlying geometry, must be refined in order to extract the novel geometrical degrees of freedom it contains. As the first option to think of, it is possible that there exist a fundamental rank (1,2) tensor field $\delta^{\lambda}_{\alpha\beta}$, and the connection $\Delta^{\lambda}_{\alpha\beta}$ equals just this fundamental tensor field. Though this is possible, at present there is no indication for such higher spin fields, and thus, it is convenient to leave this possibility aside. The other option to think of is that $\Delta^{\lambda}_{\alpha\beta}$ could be made up of lower spin fields, *i. e.* vectors, spinors and scalars. To this end, given its rank (1,2) nature, it is obvious that the tensorial connection must be decomposable into vector fields, which might be fundamental fields or composites formed out of spinors or scalars. In general, $\Delta^{\lambda}_{\alpha\beta}$ possesses 64 independent elements, and hence, it should be fully parameterizable by 3 independent vector fields, whose nature will be further analyzed in the sequel. One of the vectors is already defined by the contraction $\mathtt{V}_{\alpha}$ in (13). The other two

$$\mathtt{U}_{\alpha} \;=\; \Delta^{\mu}_{\mu\alpha} \tag{16}$$

and

$$\mathtt{W}^{\alpha} \;=\; g^{\mu\nu}\Delta^{\alpha}_{\mu\nu} \tag{17}$$

are conveniently defined through the remaining two distinct contractions of $\Delta^{\lambda}_{\alpha\beta}$. These two vectors, unlike $\mathtt{V}_{\alpha}$, do not possess an immediate kinetic term, and if they are to have any, it must come from the invariants involving the gradients of the fundamental tensors in (7).

Given the metric tensor $g_{\alpha\beta}$, $\mathtt{V}_{\alpha}$ in (13), $\mathtt{U}_{\alpha}$ in (16), and $\mathtt{W}_{\alpha}$ in (17), the tensorial connection $\Delta^{\lambda}_{\alpha\beta}$ can be algebraically decomposed as

$$\begin{aligned}\Delta^{\lambda}_{\alpha\beta} \;&=\; \delta^{\lambda}_{\alpha\beta} + a_v \mathtt{V}^{\lambda} g_{\alpha\beta} + b_v \mathtt{V}_{\alpha}\delta^{\lambda}_{\beta} + c_v \delta^{\lambda}_{\alpha}\mathtt{V}_{\beta}\\ &+\; a_u \mathtt{U}^{\lambda} g_{\alpha\beta} + b_u \mathtt{U}_{\alpha}\delta^{\lambda}_{\beta} + c_u \delta^{\lambda}_{\alpha}\mathtt{U}_{\beta}\\ &+\; a_w \mathtt{W}^{\lambda} g_{\alpha\beta} + b_w \mathtt{W}_{\alpha}\delta^{\lambda}_{\beta} + c_w \delta^{\lambda}_{\alpha}\mathtt{W}_{\beta}\\ &+\; \frac{1}{M^2}\sum\left(\nu_{xy}\mathtt{V}^{\lambda} + \upsilon_{xy}\mathtt{U}^{\lambda} + \omega_{xy}\mathtt{W}^{\lambda}\right)\mathtt{X}_{\alpha}\mathtt{Y}_{\beta}\end{aligned} \tag{18}$$

because of its higher spin assuming that a fundamental rank (1,2) tensor field $\delta^{\lambda}_{\alpha\beta}$ does not exist at all. The sum in the last line runs over $\mathtt{X},\mathtt{Y} = \mathtt{V},\mathtt{U},\mathtt{W}$, and M is a mass scale expected to be around the fundamental scale of gravity, M_{Pl}. The decomposition necessarily involves linear and trilinear combinations of the vectors. There cannot exist any other acceptable combinations of the vectors. The expansion is unique in structure. However, one notices that all three defining relations (13), (16), (17) are algebraic in nature, and thus, the dimensionless coefficients a's, …, ω's cannot be prohibited to involve dressing factors of the form $\mathtt{I}^{\delta}/M^{\delta}$ where $\delta \geq 0$ and $\mathtt{I}$ is an invariant generated by bilinear contractions of the vectors $\mathtt{V}$, $\mathtt{U}$, $\mathtt{W}$. These dressing factors introduce invariants with higher and higher mass dimension. The defining relations (13), (16) and (17) are too few to determine all the expansion coefficients in (18). Therefore, all one can do is to express nine of the coefficients in terms of the rest. For instance, the coefficients in the linear sector can be expressed in terms of those in the trilinear sector, leaving ν's, υ's and ω's undetermined, and accordingly, all the invariants in (7) can be expanded via (18) to determine the dynamics of $\mathtt{V}_{\alpha}$, $\mathtt{U}_{\alpha}$, and $\mathtt{W}_{\alpha}$. Nevertheless, as clearly suggested by (18), the main effect of trilinear terms is to generate quartic and higher order interactions of vectors. Putting emphasis on quadratic interactions, the trilinear terms can thus be left aside though they can be straightforwardly included in the formulae below by processing the complete $\Delta^{\lambda}_{\alpha\beta}$ in (18). Proceeding thus with linear terms in (18), one finds

$$\begin{aligned}a_v \;&=\; c_v = a_u = b_u = b_w = c_w = -\frac{1}{18}\\ b_v \;&=\; c_u = a_w = \frac{5}{18}\end{aligned} \tag{19}$$

for which $\Delta^{\lambda}_{\alpha\beta}$ gets decomposed linearly in terms of $\mathtt{V}_{\alpha}$, $\mathtt{U}_{\alpha}$ and $\mathtt{W}_{\alpha}$.

Given the decomposition in (18) of the tensorial connection, all the invariants in (7) can be expressed in terms of $\mathtt{V}_{\alpha}$, $\mathtt{U}_{\alpha}$ and $\mathtt{W}_{\alpha}$ to determine their dynamics as vector fields hidden in the non-Riemannian geometry under consideration. To start with, the curvature scalar $\mathbb{R}(g,\between)$, as follows from (9), is composed of the GR part $R(g,\Gamma)$ and the quasi curvature scalar $g^{\alpha\beta}\mathcal{R}_{\alpha\beta}(\Delta) \equiv \mathcal{R}(g,\Delta)$. In response to the linear part of the decomposition of $\Delta^{\lambda}_{\alpha\beta}$ in (18), the latter takes the form

$$\begin{aligned}\mathcal{R}(g,\Delta) \;&=\; \nabla\cdot(\mathtt{W}-\mathtt{U}) + \frac{1}{18}\Big(\mathtt{V}\cdot\mathtt{V} + \mathtt{U}\cdot\mathtt{U} + \mathtt{W}\cdot\mathtt{W}\\ &-\; 4\mathtt{V}\cdot\mathtt{U} - 4\mathtt{V}\cdot\mathtt{W} + 14\mathtt{U}\cdot\mathtt{W}\Big)\end{aligned} \tag{20}$$

which shows that a term linear in $\mathbb{R}(g,\between)$ in the gravitational Lagrangian yields the Einstein-Hilbert term $R(g,\Gamma)$ in GR plus a theory of three vector fields in which each vector develops a 'mass term' and mixes with the others quadratically. The vectors do not acquire a kinetic term from $\mathbb{R}(g,\between)$ since the first term at the right-hand

side of (20), the divergence of $\mathtt{W}_\alpha - \mathtt{U}_\alpha$, does not contribute to dynamics as it can be integrated out of the action by using $\sqrt{-g}\nabla\cdot(\mathtt{W}-\mathtt{U}) = \partial_\alpha\left(\sqrt{-g}\left(\mathtt{W}^\alpha - \mathtt{U}^\alpha\right)\right)$. One, however, notices that this term becomes important in higher curvature terms like $\mathbb{R}^2\left(g, \between\right)$.

From (11) it is already known that $\overline{\mathbb{R}}_{\alpha\beta}\left(\between\right)$ is the field strength tensor of the vector field $\mathtt{V}_\alpha$. Then the associated invariant in (7) becomes

$$\overline{\mathbb{R}}\bullet\overline{\mathbb{R}} = \mathtt{V}^{(-)\alpha\beta}\mathtt{V}^{(-)}_{\alpha\beta} \tag{21}$$

which is nothing but the kinetic term of the Abelian vector $\mathtt{V}_\alpha$.

Corresponding to the decomposition in (18), the torsion and non-metricity tensors take the explicit form

$$\begin{aligned}\mathbb{S}^{\lambda}_{\alpha\beta} &= \frac{1}{3}\left(\mathtt{V}_\alpha\delta^\lambda_\beta - \delta^\lambda_\alpha\mathtt{V}_\beta\right) - \frac{1}{3}\left(\mathtt{U}_\alpha\delta^\lambda_\beta - \delta^\lambda_\alpha\mathtt{U}_\beta\right), \quad (22)\\ \mathbb{Q}^{\alpha\beta}_{\lambda} &= \frac{1}{9}\Big(5\mathtt{V}_\lambda g^{\alpha\beta} - \mathtt{V}^\alpha\delta^\beta_\lambda - \delta^\alpha_\lambda\mathtt{V}^\beta\\ &- \mathtt{U}_\lambda g^{\alpha\beta} + 2\mathtt{U}^\alpha\delta^\beta_\lambda + 2\delta^\alpha_\lambda\mathtt{U}^\beta\\ &- \mathtt{W}_\lambda g^{\alpha\beta} + 2\mathtt{W}^\alpha\delta^\beta_\lambda + 2\delta^\alpha_\lambda\mathtt{W}^\beta\Big), \quad (23)\end{aligned}$$

and thus, the related invariants in (7) read as

$$\begin{aligned}\mathbb{S}\bullet\mathbb{S} &= 2\left(\mathtt{V}\cdot\mathtt{V} + \mathtt{U}\cdot\mathtt{U} - 2\mathtt{V}\cdot\mathtt{U}\right), \quad (24)\\ \mathbb{Q}\bullet\mathbb{Q} &= \frac{2}{9}\Big(22\mathtt{V}\cdot\mathtt{V} + 7\mathtt{U}\cdot\mathtt{U} + 7\mathtt{W}\cdot\mathtt{W} + 20\mathtt{V}\cdot\mathtt{U}\\ &+ 20\mathtt{V}\cdot\mathtt{W} + 14\mathtt{U}\cdot\mathtt{W}\Big), \quad (25)\\ \mathbb{Q}\bullet\mathbb{S} &= \frac{4}{3}\Big(2\mathtt{V}\cdot\mathtt{V} + \mathtt{U}\cdot\mathtt{U} - 3\mathtt{V}\cdot\mathtt{U} - \mathtt{V}\cdot\mathtt{W}\\ &+ \mathtt{U}\cdot\mathtt{W}\Big). \quad (26)\end{aligned}$$

This completes the decomposition of the quadratic invariants of the vector fields as generated by the curvature, torsion and non-metricity tensors. It is clear that these invariants provide a kinetic term only for $\mathtt{V}_\alpha$; the other two vectors, $\mathtt{U}_\alpha$ and $\mathtt{W}_\alpha$, acquire no kinetic term from any of the invariants in (7). Nevertheless, a short glance at (22) and (23) immediately reveals that the invariants formed by the gradients of curvature, torsion and non-metricity tensors can generate the requisite kinetic terms. Specifically, from (22) it is found that

$$\mathbb{D}_{\alpha\beta} = \nabla^{\between}_{\lambda}\mathbb{S}^{\lambda}_{\alpha\beta} \supset -\frac{1}{3}\mathtt{V}^{(-)}_{\alpha\beta} + \frac{1}{3}\mathtt{U}^{(-)}_{\alpha\beta} \tag{27}$$

where the terms $\mathcal{O}\left(\Delta^2\right)$ are suppressed on the basis of unnecessity. The first term at the right-hand side is the field strength tensor of $\mathtt{V}_\alpha$ as mentioned in (11) and (12). The second term is new in that it is the field strength tensor of the $\mathtt{U}_\alpha$ field. Therefore, divergence of torsion tensor generates the requisite kinetic term for $\mathtt{U}_\alpha$, and the associated invariant

$$\begin{aligned}\mathbb{D}\bullet\mathbb{D} \supset &\ \frac{1}{9}\Big(\mathtt{V}^{(-)\alpha\beta}\mathtt{V}^{(-)}_{\alpha\beta} + \mathtt{U}^{(-)\alpha\beta}\mathtt{U}^{(-)}_{\alpha\beta}\\ &- 2\mathtt{V}^{(-)\alpha\beta}\mathtt{U}^{(-)}_{\alpha\beta}\Big) \quad (28)\end{aligned}$$

encodes the kinetic terms of $\mathtt{V}_\alpha$ and $\mathtt{U}_\alpha$ as well as their kinetic mixing. One notices that, not only the divergence operation (27) but also

$$g^{\rho\alpha}\nabla^{\between}_{\rho}\mathbb{S}^{\lambda}_{\alpha\beta} = -g^{\rho\alpha}\nabla^{\between}_{\rho}\mathbb{S}^{\lambda}_{\beta\alpha} \tag{29}$$

give contributions to the kinetic terms of vectors with similar structures as (28).

The candidate kinetic terms of $\mathtt{V}_\alpha$ in (21), and the kinetic term of $\mathtt{U}_\alpha$ in (28) are of the form expected of an $U(1)$ invariance. Of course, such an invariance is explicitly broken by the 'mass terms' generated by curvature, torsion and non-metricity tensors. This is not the whole story, however. The kinetic terms generated by the derivatives of the non-metricity tensor in (23) also violate possible $U(1)$ invariance suggested by (21) and (28). To see this, one notes that

$$\begin{aligned}\mathbb{N}^{\alpha\beta} = g^{\rho\lambda}\nabla^{\between}_{\rho}\mathbb{Q}^{\alpha\beta}_{\lambda} \supset &\ \frac{1}{9}\Big(5\nabla\cdot\mathtt{V}g^{\alpha\beta} - \mathtt{V}^{(+)\alpha\beta}\\ &- \nabla\cdot\mathtt{U}g^{\alpha\beta} + 2\mathtt{U}^{(+)\alpha\beta}\\ &- \nabla\cdot\mathtt{W}g^{\alpha\beta} + 2\mathtt{W}^{(+)\alpha\beta}\Big) \quad (30)\end{aligned}$$

where

$$\mathtt{V}^{(+)}_{\alpha\beta} \equiv \nabla_\alpha\mathtt{V}_\beta + \nabla_\beta\mathtt{V}_\alpha \tag{31}$$

is the symmetric counterpart of the anti-symmetric field strength tensor $\mathtt{V}^{(-)}_{\alpha\beta}$ in (12). This definition holds also for the other vectors. Then the invariant generated by (30) reads as

$$\mathbb{N}\bullet\mathbb{N} \supset \frac{1}{162}\sum_{i,j=1}^{3}\mathtt{A}^{(+)}_{i\alpha\beta}\mathtt{K}^{\alpha\beta\mu\nu}_{ij}\mathtt{A}^{(+)}_{j\mu\nu} \tag{32}$$

where $\mathtt{A}_i \in (\mathtt{V},\mathtt{U},\mathtt{W})$, and $\mathtt{K}^{\alpha\beta\mu\nu}_{ij}$ is the (i,j)-th entry of the matrix-valued tensor

$$\mathtt{K}^{\alpha\beta\mu\nu} = \begin{pmatrix} 202g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} & g^{\alpha\beta}g^{\mu\nu} - 2g^{\alpha\mu}g^{\beta\nu} - 2g^{\alpha\nu}g^{\beta\mu} & g^{\alpha\beta}g^{\mu\nu} - 2g^{\alpha\mu}g^{\beta\nu} - 2g^{\alpha\nu}g^{\beta\mu} \\ g^{\alpha\beta}g^{\mu\nu} - 2g^{\alpha\mu}g^{\beta\nu} - 2g^{\alpha\nu}g^{\beta\mu} & -2g^{\alpha\beta}g^{\mu\nu} + 4g^{\alpha\mu}g^{\beta\nu} + 4g^{\alpha\nu}g^{\beta\mu} & -2g^{\alpha\beta}g^{\mu\nu} + 4g^{\alpha\mu}g^{\beta\nu} + 4g^{\alpha\nu}g^{\beta\mu} \\ g^{\alpha\beta}g^{\mu\nu} - 2g^{\alpha\mu}g^{\beta\nu} - 2g^{\alpha\nu}g^{\beta\mu} & -2g^{\alpha\beta}g^{\mu\nu} + 4g^{\alpha\mu}g^{\beta\nu} + 4g^{\alpha\nu}g^{\beta\mu} & -2g^{\alpha\beta}g^{\mu\nu} + 4g^{\alpha\mu}g^{\beta\nu} + 4g^{\alpha\nu}g^{\beta\mu} \end{pmatrix} \quad (33)$$

which describes the kinetic mixing among the three vector fields. As for the divergence of torsion in (28), one notices that, not only the divergence operation (30) but also

$$\nabla^{\mathbb{\Gamma}}_{\alpha}\mathbb{Q}^{\alpha\beta}_{\lambda} = \nabla^{\mathbb{\Gamma}}_{\alpha}\mathbb{Q}^{\beta\alpha}_{\lambda} \quad (34)$$

give contributions similar to that in (32). In addition to these, contraction of

$$\nabla^{\mathbb{\Gamma}}\mathbb{Q} \bullet \nabla^{\mathbb{\Gamma}}\mathbb{S} \;=\; 0 \quad (35)$$

due to symmetry conditions.

Having done with the decomposition of various invariants in terms of the vector fields $\mathtt{V}$, $\mathtt{U}$ and $\mathtt{W}$, we now turn to analysis of interactions in such a non-Riemannian setup. The most general action functional describing 'gravity' and 'matter' is of the form

$$I = \int d^4x\, \sqrt{-g}\left\{\mathfrak{L}\left(\mathbb{R}, \overline{\mathbb{R}}, \mathbb{S}, \mathbb{Q}\right) + L_m\left(\psi, g\right) - V_0\right\} \quad (36)$$

which contains action densities for geometric and material parts, respectively. V_0 stands for the vacuum energy (containing the bare cosmological term fed by the geometrical sector), and ψ stands for matter and radiation fields, collectively. Neither the geometrical $\mathfrak{L}$ nor the matter Lagrangian L_m contains any constant energy density; all such energy components are collected in V_0. The geometrical part reads explicitly as

$$\begin{aligned} \mathfrak{L} \;&=\; \frac{1}{2}M_{Pl}^2\left(\mathbb{R} + c_S\mathbb{S}\bullet\mathbb{S} + c_Q\mathbb{Q}\bullet\mathbb{Q} + c_{QS}\mathbb{Q}\bullet\mathbb{S}\right) \\ &+\; c'_S\nabla^{\mathbb{\Gamma}}\mathbb{S}\bullet\nabla^{\mathbb{\Gamma}}\mathbb{S} + c'_Q\nabla^{\mathbb{\Gamma}}\mathbb{Q}\bullet\nabla^{\mathbb{\Gamma}}\mathbb{Q} + c'_{QS}\nabla^{\mathbb{\Gamma}}\mathbb{Q}\bullet\nabla^{\mathbb{\Gamma}}\mathbb{S} \\ &+\; c_{R^2}\mathbb{R}^2 + c_{RR}\mathbb{R}\bullet\mathbb{R} + c_{\overline{RR}}\overline{\mathbb{R}}\bullet\overline{\mathbb{R}} + \mathcal{O}\left(\frac{1}{M_{Pl}^2}\right) \end{aligned} \quad (37)$$

where we have discarded terms $\mathcal{O}\left(1/M_{Pl}^2\right)$. Moreover, we have discarded higher-derivative terms $\Box^{\mathbb{\Gamma}}\mathbb{R}$ and the like. c's are all dimensionless couplings. The mass dimension-2 terms are naturally scaled by the fundamental scale of gravity, M_{Pl}. One notices that $\mathbb{R}^2$ and $\mathbb{R}\bullet\mathbb{R}$ contain higher-curvature terms $R(g,\Gamma)^2$ and $R_{\alpha\beta}(\Gamma)R^{\alpha\beta}(\Gamma)$, respectively. These terms should be absent (c_{R^2} and c_{RR} must vanish) if such ghosty higher-derivative interactions in GR are to be avoided. The remaining terms, after using their decompositions in terms of the vector fields $\mathtt{V}$, $\mathtt{U}$ and $\mathtt{W}$, give rise to the action

$$\begin{aligned} I \;&=\; \int d^4x\, \sqrt{-g}\left\{\frac{1}{2}M_{Pl}^2 R + L_m\left(\psi, g\right) - V_0\right\} \\ &+\; \int d^4x\, \sqrt{-g}\Big\{c_{VV}\mathtt{V}^{(-)\alpha\beta}\mathtt{V}^{(-)}_{\alpha\beta} + c_{UU}\mathtt{U}^{(-)\alpha\beta}\mathtt{U}^{(-)}_{\alpha\beta} + c_{VU}\mathtt{V}^{(-)\alpha\beta}\mathtt{U}^{(-)}_{\alpha\beta} \\ &+\; \mathtt{V}^{(+)}_{\alpha\beta}\mathtt{k}^{\alpha\beta\mu\nu}_{VV}\mathtt{V}^{(+)}_{\mu\nu} + \mathtt{U}^{(+)}_{\alpha\beta}\mathtt{k}^{\alpha\beta\mu\nu}_{UU}\mathtt{U}^{(+)}_{\mu\nu} + \mathtt{W}^{(+)}_{\alpha\beta}\mathtt{k}^{\alpha\beta\mu\nu}_{WW}\mathtt{W}^{(+)}_{\mu\nu} + \mathtt{V}^{(+)}_{\alpha\beta}\mathtt{k}^{\alpha\beta\mu\nu}_{VU}\mathtt{U}^{(+)}_{\mu\nu} + \mathtt{V}^{(+)}_{\alpha\beta}\mathtt{k}^{\alpha\beta\mu\nu}_{VW}\mathtt{W}^{(+)}_{\mu\nu} + \mathtt{U}^{(+)}_{\alpha\beta}\mathtt{k}^{\alpha\beta\mu\nu}_{UW}\mathtt{W}^{(+)}_{\mu\nu} \\ &+\; M_{Pl}^2\Big(\frac{1}{2}a_{VV}\mathtt{V}^{\alpha}\mathtt{V}_{\alpha} + \frac{1}{2}a_{UU}\mathtt{U}^{\alpha}\mathtt{U}_{\alpha} + \frac{1}{2}a_{WW}\mathtt{W}^{\alpha}\mathtt{W}_{\alpha} + a_{VU}\mathtt{V}^{\alpha}\mathtt{U}_{\alpha} + a_{VW}\mathtt{V}^{\alpha}\mathtt{W}_{\alpha} + a_{UW}\mathtt{U}^{\alpha}\mathtt{W}_{\alpha}\Big)\Big\} \end{aligned} \quad (38)$$

where the first integral at the right-hand side is precisely the Einstein-Hilbert action in GR (plus the contribution of matter and radiation), and the second integral pertains to a theory of three vector fields in a spacetime with metric $g_{\alpha\beta}$. The Einstein-Hilbert action above would receive contributions from higher-curvature (and thus typically ghosty) terms had we kept $\mathbb{R}^2$ and $\mathbb{R}\bullet\mathbb{R}$ terms in (37).

In essence, under the decomposition in (18), the non-Riemannian gravitational theory in (37) reduces to a tensor-vector theory of the type in (38) (leaving aside the matter sector $L_m(\psi, g)$). The vector part of the action is written in a rather generic form by admitting that various terms listed above plus similar ones coming, for example, from (29) and (34) give rise to, at the quadratic level, the structures in (38) with dimensionless coefficients $c_{VV}, \dots, a_{UW}$. These coefficients can be expressed as linear combinations of the coefficients weighing individual contributions.

The tensor-vector theory in (38) has been obtained for a general setup involving curvature, torsion and non-metricity tensors exhaustively. The theory is GR plus a theory of three vectors $\mathtt{V}$, $\mathtt{U}$ and $\mathtt{W}$. Any constraint or selection rule imposed on the non-Riemannian geometry results in a more restricted theory. It could thus be useful to discuss certain aspects of (38) here:

- Theory consists of three vector fields $\mathtt{V}$, $\mathtt{U}$ and $\mathtt{W}$. The vector action contains two types of kinetic terms: ones with $\mathtt{X}^{(-)}_{\alpha\beta}$ and those with $\mathtt{X}^{(+)}_{\alpha\beta}$. The $\mathtt{V}$ and $\mathtt{U}$ possess both types of kinetic terms while $\mathtt{W}$ possesses only the second type *i. e.* $\mathtt{W}^{(+)}_{\alpha\beta}$. The $\mathtt{X}^{(-)}_{\alpha\beta}$ and hence the corresponding kinetic terms obviously possess an Abelian invariance. However, there is no such invariance for the kinetic terms involving $\mathtt{X}^{(+)}_{\alpha\beta}$. Therefore, the vector fields contained in (38) are not associated with a gauge theory; they are not vectors originating from need to realize a local U(1) invariance.

  The coefficients $c_{VV}, \dots, k^{\alpha\beta\mu\nu}_{UV}$, which seem being left arbitrary, can actually be fixed in terms of the coefficients of individual terms in (37) which contribute to that particular structure. The kinetic terms, both $\mathtt{X}^{(-)}_{\alpha\beta}$ and $\mathtt{X}^{(+)}_{\alpha\beta}$ type, receive contributions from various structures, as addressed before. In particular, contributions of the alternative structures given in (29) and (34) must also be included in forming the vector action in (38).

- A highly crucial aspect concerns the signs of the coefficients $c_{VV}, \dots, k^{\alpha\beta\mu\nu}_{UV}$ in the kinetic part of the vector action. The kinetic terms of $\mathtt{V}$, $\mathtt{U}$ and $\mathtt{W}$ must have the correct sign required of a ghost-free theory. Indeed, any sign-flip in the kinetic terms causes vector ghosts to show up in the spectrum. The various coefficients in (37) must comply with this requirement.

- The vectors exhibit not only the kinetic mixings $\mathtt{X}^{(-)}_{\alpha\beta}\mathtt{Y}^{(-)\alpha\beta}$ and $\mathtt{X}^{(+)}_{\alpha\beta}\mathtt{Y}^{(+)\alpha\beta}$ but also mass mixings of the form $\mathtt{X}^{\alpha}\mathtt{Y}_{\alpha}$, as shown in the last line of the vector action. Their masses and mixings are proportional to M_{Pl} with respective coefficients $a_{VV}, \dots, a_{UW}$. In $\{\mathtt{V}, \mathtt{U}, \mathtt{W}\}$ basis their mass-squared matrix reads as

$$\frac{1}{2} M_{Pl}^2 \begin{pmatrix} a_{VV} & a_{VU} & a_{VW} \\ a_{VU} & a_{UU} & a_{UW} \\ a_{VW} & a_{UW} & a_{WW} \end{pmatrix} \tag{39}$$

  each entry of which can be extracted from (37) as

$$\begin{aligned} a_{VV} &= \frac{1}{18} + 2c_S + \frac{44}{9}c_Q + \frac{8}{3}c_{QS}\,, \\ a_{UU} &= c_{WW} + 2c_S + \frac{4}{3}c_{QS}\,, \\ a_{WW} &= \frac{1}{18} + \frac{14}{9}c_Q\,, \\ a_{VU} &= -\frac{1}{9} - 2c_S + \frac{20}{9}c_Q - 2c_{QS}\,, \\ a_{VW} &= -\frac{1}{9} + \frac{20}{9}c_Q - \frac{2}{3}c_{QS}\,, \\ a_{UW} &= \frac{7}{18} + \frac{14}{9}c_Q + \frac{2}{3}c_{QS}\,. \end{aligned} \tag{40}$$

  It is the eigenvalues of (39) that determine the light and heavy vector spectrum in the theory. If off-diagonal entries are small *i. e.* if c_S, c_Q and c_{QS} are chosen appropriately then all three vector bosons weigh $M_{Pl}/3\sqrt{2}$. Alternatively, if the mixings are sizeable, or equivalently, if all entries of (39) are of similar size then there will exist two light and one heavy vectors in the spectrum. Depending on the hierarchy of the couplings, there could exist just one light state instead of two [4]. In any case, it is with the hierarchy of the couplings that the vector boson spectrum can exhibit different hierarchies. Needless to say, the intra-hierarchy of the mass matrix entries $a_{VV}, \dots, a_{UW}$ is determined by the couplings c_S, c_Q and c_{QS} via the relations (40).

  Actually, having the vector fields with masses around M_{Pl} should come by no surprise; the underlying theory (37) is a pure gravity of non-Riemannian structure, and the mass scale in the theory is automatically fixed by the fundamental scale of gravity M_{Pl}. However, the statement 'a Planckian-mass vector field' depends crucially on what we mean by the vector field: Is it fundamental or is it a composite structure? We will discuss answers and consequences of these questions in the sequel.

- The framework we have reached in (38) is a rather general one in that we have imposed no condition on metric, connection and any other geometro-dynamical quantity. Imposition of certain selection rules, though seems to cause loss of generality, does actually prove highly useful for extracting information about behavior of the system in certain reasonable situations. Here we shall discuss two such limiting cases:

  - **Symmetric Connection:** We first discuss the possibility of symmetric connection *i. e.* $\between^{\lambda}_{\alpha\beta} = \between^{\lambda}_{\beta\alpha}$. The prime implication of this selection rule is that the torsion tensor identically

vanishes, $\mathbb{S}^{\lambda}_{\alpha\beta} = 0$. This statement is equivalent to imposing

$$\mathtt{V}_\alpha = \mathtt{U}_\alpha, \tag{41}$$

as is manifestly suggested by the decomposition of $\Delta^{\lambda}_{\alpha\beta}$ in (18). This constraint is seen to nullify the invariants $\mathbb{S}\cdot\mathbb{S}$ and $\mathbb{S}\cdot\mathbb{Q}$, in agreement with vanishing torsion. This particular relation between $\mathtt{V}$ and $\mathtt{U}$ reduces the vector action in (38) into a theory of two vectors: the $\mathtt{V}$ and $\mathtt{W}$. The structure remains similar to that in (38) yet various terms containing $\mathtt{V}$ and $\mathtt{U}$ merge together to give more compact relations.

– **Antisymmetric Tensorial Connection:** This time we consider the relation $\Delta^{\lambda}_{\alpha\beta} = -\Delta^{\lambda}_{\alpha\beta}$ for the tensorial connection not for $\between^{\lambda}_{\alpha\beta}$. Actually, since $\Gamma^{\lambda}_{\alpha\beta}$ is symmetric the connection $\between^{\lambda}_{\alpha\beta}$ possesses no obvious symmetry under the exchange of α and β. The prime implication of the anti-symmetric Δ is that the geodesics of test bodies remain as in the GR. This, however, does not mean that one can eliminate the non-Riemannian effects. The reason is that the geodesic deviation, which involves the Riemann tensor $\mathbb{R}^{\alpha}_{\mu\beta\nu}$, directly feels the non-GR components of the curvature tensor. In the language of the expansion (18), anti-symmetric $\Delta^{\lambda}_{\alpha\beta}$ gives

$$\mathtt{V}_\alpha = -\mathtt{U}_\alpha \quad \text{and} \quad \mathtt{W}_\alpha = 0 \tag{42}$$

which reduces thus the vector action in (38) to theory of a single vector field $\mathtt{V}$.

Here we have highlighted certain salient features of the Tensor-Vector theory of (38) in regard to various structures and limiting cases the vector part can take.

APPLICATIONS TO COSMOLOGY

Up to now, we have constructed a general action which consists of all possible vector and tensor fields. In addition to this, we have given two limiting cases as symmetric and antisymmetric tensorial connection. In next two subsections, by using antisymmetric tensorial connection limit and some constraints, we obtain two well-known actions which are defined in modified gravity theories These are TeVeS gravity and Vector Inflation.

TeVeS Gravity

In spite of its great success in describing the solar system, General Relativity (GR) fails to account for dynamics at galactic scales without postulating a large amount of cold dark matter (CDM) – non-baryonic, non-relativistic, electrically neutral, weakly interacting particles of weak-scale masses [5]. The asymptotic flatness of the galaxy rotation curves, which occurs towards galaxy outskirts involving extremely small accelerations, manifestly disagrees with predictions of the GR unless the galactic region is populated by non-shining, and hence, astrophysically unobservable CDM.

Apart from this, there are problems with structure formation: with the baryonic matter alone, the large-scale structure as we observe it would not have been formed yet if gravity is described by GR. Indeed, GR demands large amounts of 'dark components' (23% 'dark matter' for structure formation and 73% 'dark energy' for late-time inflation) to be able to account for the mounting cosmo-physical precision data (coming from observations on microwave background [6], large scale structure [7], and supernovae [8]). However, the way these dark components enter into gravitational field equations does not involve their origins and nature; they are treated as 'fluids' with right density and equation of state. Nevertheless, the positron excess reported by recent observations [9] on cosmic rays, if interpreted to come from decays or annihilations of dark matter, can be taken as indirect signals of dark matter (though there are alternative arguments in favor of astrophysical sources [10] of positron excess).

This 'dark paradigm' necessitated by GR can in fact be evaded if an alternative description of Nature takes over at extremely small accelerations and curvatures. This is what has been postulated by Milgrom [11], who replaced Newton's second law of motion with

$$\mu\left(\frac{|\vec{a}|}{a_0}\right)\vec{a} = -\vec{\nabla}\Phi_N \tag{43}$$

where Φ_N is the gravitational potential, $\mu(x) \rightsquigarrow 1(x)$ for $x \gg 1 (x \ll 1)$, and $a_0 \simeq 10^{-10}\ m/s^2$ is an acceleration scale appropriate for galaxy outskirts [12]. This proposal, despite its empirical success, had to wait for the relativistic generalizations of [13, 14] to become a complete, alternative theory of gravitational interactions (see also the review [15]). The relativistic generalization, dubbed as tensor-vector-scalar (TeVeS) theory of gravity, involves the *geometrical* fields V_μ and ϕ in addition to the metric tensor $g_{\mu\nu}$ such that, while the matter sector involves $g_{\mu\nu}$ only, the gravitational sector involves

$$\widetilde{g}_{\mu\nu} = e^{2\phi}g_{\mu\nu} - 2\sinh(2\phi)A_\mu A_\nu \tag{44}$$

whose action can be generalized to incorporate aether effects [16], too. Various astrophysical and cosmological phenomena exhibit observable signatures of TeVeS [17].

TeVeS is essentially a bi-metrical gravitational theory where matter and gravity are distinguished by the metric fields they operate with. It is thus natural to expect a reformulation in bi-metrical language [18] wherein certain interactions and properties follow deductively.

The material produced in the last section is general and detailed enough to have a re-look at the TeVeS gravity. In this section we will argue that TeVeS type extended gravity theories do naturally follow from the non-Riemannian theories of the form (37) under the decomposition (18).

- To establish contact with TeVeS gravity, it is necessary to discuss first the function μ defined in (43). In relativistic formulation, μ is a non-dynamical field in the action. Variation of the action with respect to μ fixes 'gradient' of its potential $dV(\mu)/d\mu$ in terms of the remaining terms in which μ appears at least linearly. Basically, μ must multiply the kinetic term of (scalars or vectors) so that its force $dV(\mu)/d\mu$ is fixed in terms of the field gradient-squareds (actually the kinetic terms of the fields) in accord with the requirements of the MOND. In summary, the MOND relation (43) for μ arises from the equation of motion for μ (to be solved via $dV(\mu)/d\mu$ in terms of the kinetic terms of the fields in the spectrum). The relativistic theory of [14] requires that μ should be non-dynamical, that is, it should have no kinetic term. Therefore, the Lagrangian of μ can be directly constructed from couplings in the action (38). We do this as follows:
  - First, we postulate that the vacuum energy density V_0 in (37) and (38) can actually be decomposed as
  $$V_0 = V(\mu) + \Delta V \tag{45}$$
  where ΔV is a constant additive energy density while V varies with μ. At this stage μ is a hypothetical parameter having no solid physical basis.
  - We further postulate that the coefficients $c_{VV}, \dots, k_{UV}^{\alpha\beta\mu\nu}$ weighing the individual kinetic terms in the vector part of the action (38) do actually depend on the parameter μ at least in a linear fashion. In fact, it is not necessary to make all these constants vary with μ; the μ dependence of one single parameter suffices.
- Under these instructed changes for 'creating' or 'explicating' the non-dynamical field μ, the action (38) becomes essentially the Tensor-Vector Theory of [16]. This theory is obtained by eliminating the scalar field through the constraints on the bimetrical theory of [14], and is shown to be a viable replacement for cold dark matter. As an aether theory, it works as good as the model in [14] as far as the MOND-change of gravity is concerned. The main distinction between the theory obtained here and that of [16] is that the model here consists of three vectors in the most general case. If one specializes to cases like (41) or (42), however, the model obtained here gets closer to the aether theory of [16], which is shown therein to be an alternative to the cold dark matter.

  Consequently, the Tensor-Vector theory in (38) provides a general enough framework (in terms of parameters and number of vector fields) for generating the TeVeS gravity of [11, 13–15] through the analyses in [16]. It should be kept in mind that, the TeVeS gravity of [13, 14] is based on a bimetrical theory where the geometrical sector proceeds with metric involving a scalar field, vector field and the metric field used by the matter Lagrangian. The theory in the present work, however, provides a compact approach to TeVeS gravity via the decomposition of the tensorial connection in (18).
- At this point, one may wonder why we are dealing with the Tensor-Vector theory of [16] instead of the true TeVeS gravity of [13–15]. Actually, as we will shown below, the action (38) naturally contains the true TeVeS gravity. To this end, the right question to ask concerns the vector fields themselves: Are they fundamental vector fields or composites of some other fields? They each could be of either nature. Whatever their structure, however, they must be true vectors on the spacetime manifold such that their vector property must not depend on the connections $\between^{\lambda}_{\alpha\beta}$ or $\Gamma^{\lambda}_{\alpha\beta}$ or $\Delta^{\lambda}_{\alpha\beta}$. The reason is that the vectors themselves are just parameterizing the connection via (18), and hence, their independence from the connection is required by the logical consistency of the construction. This constraint prohibits all structures but
  $$\mathtt{V}_\alpha = a_1 V_\alpha + \frac{a_0}{M_{Pl}} \partial_\alpha \phi \tag{46}$$
  where V_α is a fundamental vector, and ϕ is a fundamental scalar field. The vector property of V_α is obvious. Why the ϕ-dependent part is a vector is guaranteed by the fact that $\nabla_\alpha^{(any\ connection)} \phi = \partial_\alpha \phi$, and hence, it is a vector on the manifold independent of the connection; may it be $\between^{\lambda}_{\alpha\beta}$ or $\Gamma^{\lambda}_{\alpha\beta}$ or some other structure. The expansion (46) is valid for each of the vectors $\mathtt{V}$, $\mathtt{U}$ and $\mathtt{W}$ with their own scalar fields in each case.

  It is obvious that replacement of (46) and similar relations for $\mathtt{U}_\alpha$ and $\mathtt{W}_\alpha$ into the vector action in (38) will yield a general tensor-vector-scalar theory of gravity. The main difference from [13–15] will be the number of vectors and scalars in the theory. The difference will be the dependence of the action on the scalars: Only the gradients of scalars are involved. The scalars themselves do not enter the action. Nevertheless, one arrives at a tensor-vector-scalar theory of gravity, and the theory is paramet-

rically and dynamically wide enough to cover the standard TeVeS gravity.

- As a concrete case study, here we shall discuss the reduced theory after imposing the condition (42). The action (38) reduces to

$$
\begin{aligned}
I &= \int d^4x\, \sqrt{-g}\Bigg\{\frac{1}{2}M_{Pl}^2 R + L_m(\psi, g) - V_0 \\
&+ \overline{c}_{VV}\mathtt{V}^{(-)\alpha\beta}\mathtt{V}^{(-)}_{\alpha\beta} + \mathtt{V}^{(+)}_{\alpha\beta}\overline{\mathtt{k}}^{\alpha\beta\mu\nu}_{VV}\mathtt{V}^{(+)}_{\mu\nu} \\
&+ \frac{1}{2}M_{Pl}^2\overline{a}_{VV}\mathtt{V}^{\alpha}\mathtt{V}_{\alpha}\Bigg\}
\end{aligned} \tag{47}
$$

where the terms involving $\mathtt{V}$ and $\mathtt{U}$ in (38) combine to form the over-lined coefficients in here. The terms involving $\mathtt{W}$ in (38) are all nullified in accord with (42). For instance, one directly finds

$$
\overline{a}_{VV} = \frac{1}{3} + 8c_S + 2c_Q + 8c_{QS} \tag{48}
$$

form (40). The reduced theory in (47) is precisely the one in [16] except for the absence of quartic-in-$\mathtt{V}$ terms. The couplings in and dynamics of the two theories can be matched via the terms involved in two cases. This situation becomes especially clear after using $\mathtt{V}^{\alpha}\mathtt{V}_{\alpha} = -1$ in the tensor-vector theory of [16].

Now, it is time to analyze (47) under the decomposition (46). One finds

$$
\begin{aligned}
I &= \int d^4x\, \sqrt{-g}\Bigg\{\frac{1}{2}M_{Pl}^2 R + L_m(\psi, g) - V_0 \\
&+ a_1^2\overline{c}_{VV}V^{(-)\alpha\beta}V^{(-)}_{\alpha\beta} + a_1^2 V^{(+)}_{\alpha\beta}\overline{\mathtt{k}}^{\alpha\beta\mu\nu}_{VV}V^{(+)}_{\mu\nu} \\
&+ \frac{1}{2}M_{Pl}^2 a_1^2\overline{a}_{VV}V^{\alpha}V_{\alpha} + M_{Pl}a_1 a_0\overline{a}_{VV}V^{\alpha}\partial_{\alpha}\phi \\
&+ a_0^2\overline{a}_{VV}\partial^{\alpha}\phi\partial_{\alpha}\phi + \mathcal{O}\left(\frac{1}{M_{Pl}}\right)\Bigg\}
\end{aligned} \tag{49}
$$

from which it is seen that setting $V_0 \equiv V(\mu) + \Delta V$ and $a_0 = \bar{a}_0\mu$ essentially suffices to reproduce the results of TeVeS gravity [14, 15]. Setting $V^{\alpha}V_{\alpha} = -1$ as a constraint on the vector field, the mass term of V^{α} in (49) just adds up to the vacuum energy V_0.

Vector Inflation

According to the standart big bang cosmology, which is defined by using Friedmann-Robertson-Walker (FRW) metric, universe is homogeneous and isotropic on large scales [23]. In addition to this, observations of Hubble in redshifts of galaxies shows that universe is expanding. To understand dynamical properties of expansion, the solutions of Einstein equation for FRW metric are required. Combination of these solutions is given by

$$
\frac{\ddot{a}}{a} = -\frac{4\pi}{3M_{pl}^2}(\rho + 3p) \tag{50}
$$

as ρ implies energy density and p is pressure and a is scale factor. In the ligth of equation (50) one can think that universe expands by decelerating in case of $(\rho + 3p) > 0$. However, this deceleration doesn't solve some problem of standart big bang cosmology such as flatness, horizon and so on. To solve these problems, accelerated expansion of universe in early stage is treated instead of decelerated one i.e $(\rho + 3p) < 0$ and this type of expansion is called "inflation". Inflation is generally driven by scalar fields to prevent anisotropy occured in higher spin fields [24]. However, scalar inflation models have fine-tuning problem and also scalar bosons which is base of these models aren't observed by experiments[25]. Therefore, vector inflation model is condsidered instead of scalar inflation model. Higher spin inflation model is also considered in literature[26].

Vector inflation was firstly proposed in [21] by using spacelike vector fields. In Ford's paper vector fields gave anisotropic solution of inflation. So, instead of spacelike vector fields, it was shown that timelike vector fields under some constraints of vector field potential give rise to desired inflationary expansion [22]. The other problems vector fields have can be solved by using a triplet of mutually orthogonal vector fields and non- minimally coupling.

In this section, we show that after a regularization ,the action (38) obtained by using the anti-symmetric connection constraint give the same action in [22] which is most general action of vector inflation theory.

Combining the abelian and non-abelian part of vector field and defining new dimensionless coefficients lead to the action

$$
I = \int d^4x\, \sqrt{-g}\Bigg\{\frac{1}{2}M_{Pl}^2 R + \frac{1}{2}\kappa^{\alpha\beta\mu\nu}\nabla_{\alpha}V_{\beta}\nabla_{\mu}V_{\nu} + V(\xi)\Bigg\} \tag{51}
$$

where

$$
V(\xi) = \frac{1}{2}M_{Pl}^2\overline{a}_{VV}V^{\alpha}V_{\alpha} \tag{52}
$$

and

$$
\kappa^{\alpha\beta\mu\nu} = \kappa_1 g^{\alpha\beta}g^{\mu\nu} + \kappa_2 g^{\alpha\mu}g^{\beta\nu} + \kappa_3 g^{\alpha\nu}g^{\beta\mu} \tag{53}
$$

$\xi = V^{\alpha}V_{\alpha}$, and $\kappa_1, \kappa_2, \kappa_3$ are random coefficients coming

from general action.

$$\begin{aligned}\kappa_1 &= \frac{44}{18}c_Q^{'},\\ \kappa_2 &= \frac{8c_s^{'}+2c_Q^{'}}{18},\\ \kappa_3 &= \frac{2c_Q^{'}-8c_s^{'}}{18}\end{aligned} \tag{54}$$

Assigning suitable values to these coefficients reproduce the same results with the action of general vector inflation in [22].

CONCLUSION

Metric-affine gravity generalizes the GR by accommodating an affine connection that extends the Levi-Civita connection. The tensorial part of the connection, under general conditions, can be decomposed into three independent vector fields (and a fundamental rank (1,2) tensor field, if any) which can be fundamental fields or gradients of some scalar fields. By this way the vector, scalar and tensor fields come into play when the metric-affine action is decomposed accordingly. The resulting theory is rather general. By imposing judicious constraints, theory can be reduced to more familiar ones like TeVeS gravity, vector inflation or aether-like models, in general. In the text we have given a detailed discussion of the TeVeS gravity and vector inflation.

From this work, one concludes that metric-affine gravity is rich enough to supply various vector and scalar fields needed in cosmological phenomena. Analyses of various effects may lead to a standard model of metric-affine gravity.

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APPENDIX

Contraction of tensors becomes a tedious operation as their rank becomes larger and larger. Already at the rank-3 level, there arise various possibilities in contracting the indices. Indeed, if one defines

$$\mathbb{A}\bullet\mathbb{B} \equiv \mathbb{A}^{\lambda}_{\alpha\beta}\Xi^{\alpha\beta\mu\nu}_{\lambda\rho}\mathbb{B}^{\rho}_{\mu\nu} \tag{55}$$

the contraction tensor $\Xi^{\alpha\beta\mu\nu}_{\lambda\rho}$ is found to have the most general form

$$\begin{aligned}\Xi^{\alpha\beta\mu\nu}_{\lambda\rho} &= g_{\lambda\rho}\left(g^{\alpha\beta}g^{\mu\nu}\oplus g^{\alpha\mu}g^{\beta\nu}\oplus g^{\alpha\nu}g^{\beta\mu}\right)\\ &\oplus \delta^{\alpha}_{\lambda}\left(g^{\beta\nu}\delta^{\mu}_{\rho}\oplus g^{\beta\mu}\delta^{\nu}_{\rho}\oplus g^{\mu\nu}\delta^{\beta}_{\rho}\right)\\ &\oplus \delta^{\beta}_{\lambda}\left(g^{\alpha\nu}\delta^{\mu}_{\rho}\oplus g^{\alpha\mu}\delta^{\nu}_{\rho}\oplus g^{\mu\nu}\delta^{\alpha}_{\rho}\right)\\ &\oplus \delta^{\mu}_{\lambda}\left(g^{\beta\nu}\delta^{\alpha}_{\rho}\oplus g^{\alpha\beta}\delta^{\nu}_{\rho}\oplus g^{\alpha\nu}\delta^{\beta}_{\rho}\right)\\ &\oplus \delta^{\nu}_{\lambda}\left(g^{\beta\mu}\delta^{\alpha}_{\rho}\oplus g^{\alpha\beta}\delta^{\mu}_{\rho}\oplus g^{\alpha\mu}\delta^{\beta}_{\rho}\right)\end{aligned} \tag{56}$$

where $\oplus$ implies $+$ or $-$ depending on whether symmetric or antisymmetric combinations of the indices are involved. Clearly, $\oplus$ also contains the appropriate symmetry factors.

As an example, let us take $\mathbb{B}^{\rho}_{\mu\nu}=\mathbb{S}^{\rho}_{\mu\nu}$ which is antisymmetric in (μ,ν). In this case, when contracting $\mathbb{S}^{\rho}_{\mu\nu}$ with $\Xi^{\alpha\beta\mu\nu}_{\lambda\rho}$ only the anti symmetric part of $\Xi^{\alpha\beta\mu\nu}_{\lambda\rho}$ in (μ,ν) matters. In other words, when $\mathbb{B}^{\rho}_{\mu\nu}=\mathbb{S}^{\rho}_{\mu\nu}$ we consider only

$$\begin{aligned}\Xi^{\alpha\beta[\mu\nu]}_{\lambda\rho} &= \frac{1}{2}\Bigg[g_{\lambda\rho}\left(g^{\alpha\mu}g^{\beta\nu}-g^{\alpha\nu}g^{\beta\mu}\right)\\ &\oplus \delta^{\alpha}_{\lambda}\left(g^{\beta\nu}\delta^{\mu}_{\rho}-g^{\beta\mu}\delta^{\nu}_{\rho}\right)\\ &\oplus \delta^{\beta}_{\lambda}\left(g^{\alpha\nu}\delta^{\mu}_{\rho}-g^{\alpha\mu}\delta^{\nu}_{\rho}\right)\\ &\oplus \Big[\delta^{\mu}_{\lambda}\left(g^{\beta\nu}\delta^{\alpha}_{\rho}\oplus g^{\alpha\beta}\delta^{\nu}_{\rho}\oplus g^{\alpha\nu}\delta^{\beta}_{\rho}\right)\\ &- \delta^{\nu}_{\lambda}\left(g^{\beta\mu}\delta^{\alpha}_{\rho}\oplus g^{\alpha\beta}\delta^{\mu}_{\rho}\oplus g^{\alpha\mu}\delta^{\beta}_{\rho}\right)\Big]\Bigg]\end{aligned} \tag{57}$$

which is anti-symmetric in (μ,ν).

If $\mathbb{A}^{\lambda}_{\alpha\beta}$ in (55) is antisymmetric in (α,β) then we consider antisymmetric part of (57).

$$\begin{aligned}\Xi^{[\alpha\beta][\mu\nu]}_{\lambda\rho} &= \frac{1}{4}\Big[\delta^{\alpha}_{\lambda}\left(g^{\beta\nu}\delta^{\mu}_{\rho}-g^{\beta\mu}\delta^{\nu}_{\rho}\right)-\delta^{\beta}_{\lambda}\left(g^{\alpha\nu}\delta^{\mu}_{\rho}-g^{\alpha\mu}\delta^{\nu}_{\rho}\right)\Big]\\ &+ \frac{1}{4}\Big[\delta^{\mu}_{\lambda}\left(g^{\beta\nu}\delta^{\alpha}_{\rho}-g^{\alpha\nu}\delta^{\beta}_{\rho}\right)-\delta^{\nu}_{\lambda}\left(g^{\beta\mu}\delta^{\alpha}_{\rho}-g^{\alpha\mu}\delta^{\beta}_{\rho}\right)\Big]\\ &+ \frac{1}{2}g_{\lambda\rho}\left(g^{\alpha\mu}g^{\beta\nu}-g^{\alpha\nu}g^{\beta\mu}\right)\end{aligned} \tag{58}$$

For instance, $\mathbb{S}\bullet\mathbb{S}$ will be computed by using this contraction tensor.

However, if $\mathbb{A}^{\lambda}_{\alpha\beta}$ in (55) is symmetric in (α,β) then we have to consider symmetric part of (57).

$$\begin{aligned}\Xi^{(\alpha\beta)[\mu\nu]}_{\lambda\rho} &= \frac{1}{4}\Big[\delta^{\alpha}_{\lambda}\left(g^{\beta\nu}\delta^{\mu}_{\rho}-g^{\beta\mu}\delta^{\nu}_{\rho}\right)+\delta^{\beta}_{\lambda}\left(g^{\alpha\nu}\delta^{\mu}_{\rho}-g^{\alpha\mu}\delta^{\nu}_{\rho}\right)\Big]\\ &+ \frac{1}{4}\Big[\delta^{\mu}_{\lambda}\left(g^{\beta\nu}\delta^{\alpha}_{\rho}+g^{\alpha\nu}\delta^{\beta}_{\rho}\right)-\delta^{\nu}_{\lambda}\left(g^{\beta\mu}\delta^{\alpha}_{\rho}+g^{\alpha\mu}\delta^{\beta}_{\rho}\right)\Big]\\ &+ \frac{1}{2}g^{\alpha\beta}\left(\delta^{\mu}_{\lambda}\delta^{\nu}_{\rho}-\delta^{\nu}_{\lambda}\delta^{\mu}_{\rho}\right)\end{aligned} \tag{59}$$

For instance, $\mathbb{Q}\bullet\mathbb{S}$ should be computed by using this contraction tensor. In computing $\mathbb{Q}\bullet\mathbb{Q}$ we should sym-

metrize in both (μ,ν) and (α,β). Then contraction tensor of $\mathbb{Q}\bullet\mathbb{Q}$ is given

$$\begin{aligned}\Xi^{(\alpha\beta)(\mu\nu)}_{\lambda\rho} &= g^{\alpha\beta}g^{\mu\nu}g_{\lambda\rho}+\frac{1}{2}\Big[g_{\lambda\rho}\left(g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\beta\mu}\right)\\ &+ g^{\mu\nu}\left(\delta^{\alpha}_{\lambda}\delta^{\beta}_{\rho}+\delta^{\beta}_{\lambda}\delta^{\alpha}_{\rho}\right)+g^{\alpha\beta}\left(\delta^{\mu}_{\lambda}\delta^{\mu}_{\rho}+\delta^{\nu}_{\lambda}\delta^{\mu}_{\rho}\right)\Big]\\ &+ \frac{1}{4}\Big[\delta^{\alpha}_{\lambda}\left(g^{\beta\nu}\delta^{\mu}_{\rho}+g^{\beta\mu}\delta^{\nu}_{\rho}\right)+\delta^{\beta}_{\lambda}\left(g^{\alpha\nu}\delta^{\mu}_{\rho}+g^{\alpha\mu}\delta^{\nu}_{\rho}\right)\Big]\\ &+ \frac{1}{4}\Big[\delta^{\mu}_{\lambda}\left(g^{\beta\nu}\delta^{\alpha}_{\rho}+g^{\alpha\nu}\delta^{\beta}_{\rho}\right)+\delta^{\nu}_{\lambda}\left(g^{\beta\mu}\delta^{\alpha}_{\rho}+g^{\alpha\mu}\delta^{\beta}_{\rho}\right)\Big]\end{aligned}\tag{60}$$

In addition to these, one can compute contraction of divergence of tensors as

$$\nabla^{\between}A\bullet\nabla^{\between}B=\nabla^{\between}_{\lambda}A^{\lambda}_{\alpha\beta}\Xi^{\alpha\beta\mu\nu}\nabla^{\between}_{\rho}B^{\rho}_{\mu\nu}\tag{61}$$

$\Xi^{\alpha\beta\mu\nu}$ is contraction tensor and defined in general form as

$$\Xi^{\alpha\beta\mu\nu}=g^{\alpha\beta}g^{\mu\nu}\oplus g^{\alpha\mu}g^{\beta\nu}\oplus g^{\alpha\nu}g^{\beta\mu}\tag{62}$$

If A is symmetric in (α,β) and B is symmetric in (μ,ν) contraction tensor takes the form

$$\Xi^{(\alpha\beta)(\mu\nu)}=g^{\alpha\beta}g^{\mu\nu}+\frac{1}{2}\left(g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\beta\mu}\right)\tag{63}$$

this contraction tensor can be used to compute $\nabla\mathbb{Q}\bullet\nabla\mathbb{Q}$ because $\mathbb{Q}$ is symmetric in $(\alpha\beta)$. To compute $\nabla\mathbb{S}\bullet\nabla\mathbb{S}$, one needs contraction tensor which is antisymmetric both couple of indices.So,

$$\Xi^{[\alpha\beta][\mu\nu]}=\frac{1}{2}\left(g^{\alpha\mu}g^{\beta\nu}-g^{\alpha\nu}g^{\beta\mu}\right)\tag{64}$$

If one writes contraction tensor of $\nabla\mathbb{Q}\bullet\nabla\mathbb{S}$, it is as;

$$\Xi^{(\alpha\beta)[\mu\nu]}=0\tag{65}$$

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